Chapter 1 Supplement:

Game Theory

Game Theory

- **Competition is an important factor in decision-making**
  - **Strategies undertaken by competition can dramatically affect the outcome of a decision**

- **Game theory is one way of considering the impact of others strategies on our own strategies and outcomes**
  - ***Game*: “a decision situation with two or more decision makers in competition to win”**
  - ***Game theory*: “the study of how optimal strategies are formulated in conflict”**

Game Theory

- **Dates back to 1944**
  - ***Theory of Games and Economic Behavior*, Von Neumann & Morgenstern**

- **Widely used, many applications**
  - **War strategies**
  - **Collective bargaining**
  - **Business**

Game Theory

- **Classifications**
  - **Number of competitive decision makers (players)**
    - » **two-person game**
    - » **n-player game**
  - **Outcome in terms of each player's gains and losses**
    - » **zero-sum game: sum of gains and losses = 0**
    - » **non-zero-sum game: sum of gains and losses ≠ 0**
  - **Number of strategies employed**
- **Examples:**
  - **A union negotiating a new contract with management**
  - **Two armies conducting a war game**
  - **A retail firm and a competitor**

Game Theory Assumptions

Two-Person

Zero Sum

Gains and losses for both players sum to zero

Example

Players *X*'s Gains = $3.00
Player *Y*'s Losses = $3.00
Sum for both players = $0.00

A Two-Person, Zero-Sum Game

There are two lighting fixture stores, *X* and *Y*, who have had relatively stable market shares. Two new marketing strategies being considered by store *X* may change this peaceful coexistence. The *payoff table* below shows the potential affects on market share if both stores begin to advertise.

Store *X* Strategies	Store *Y* Strategies: 1 (radio)	Store *Y* Strategies: 2 (newspaper)
1 (radio)	2	7
2 (newspaper)	6	-4

(player trying to maximize the game outcome is on left, player trying to minimize the game outcome is on top)

Two-Person, Zero-Sum Game

Store *X* Strategies	Store *Y* Strategies: 1 (radio)	Store *Y* Strategies: 2 (newspaper)
1 (radio)	2	7
2 (newspaper)	6	-4

- **Assumptions: payoff table is known to all players**
- **Definitions:**
  - ***Strategy*: a plan of action to be followed by a player**
  - ***Value of the game*: the offensive player's gain and the defensive player's loss (in a zero-sum game)**
    - **if Store *X* selects strategy 2 & Store *Y* selects strategy 1, the outcome is a 6% gain in market share for Store *X* and a 6% loss for Store *Y***
    - **Purpose of the Game: to select the strategy resulting in the best possible outcome regardless of what the opponent does (i.e., *the optimal strategy*)**

A Pure Strategy Game

- **Each player adopts a single strategy as an optimal strategy**
  - **Strategies each player follows will always be the same irrespective of the other player's strategy**

- **Can be solved according to the *minimax* decision criterion**
  - **Each player seeks to minimize the maximum possible loss or maximize the minimum possible gain**
    - » **offensive player selects the strategy with the largest of the minimum payoffs (maximin)**
    - » **defensive player selects the strategy with the smallest of the maximum payoffs (minimax)**

A Pure Strategy Game

- Maximin strategy for Store *X*, the ***offensive*** player
  - Optimal strategy is strategy 1

Store *X* Strategies	Store *Y* Strategies: 1 (radio)	Store *Y* Strategies: 2 (newspaper)	Minimum Payoff
1 (radio)	3	5	3
2 (newspaper)	1	-2	-2

row minimums

maximin: maximum of the minimum payoffs

A Pure Strategy Game

- **Minimax strategy for Store *Y*, the *defensive* player**
  - **Optimal strategy is strategy 1**

Store *X* Strategies	**Store *Y* Strategies**	
	1 (radio)	**2 (newspaper)**
1 (radio)	**3**	**5**
2 (newspaper)	**1**	**-2**
Maximum Payoff	**3**	**5**

column maximums

minimax: minimum of the maximum payoffs

A Pure Strategy Game

- **Optimal strategy for each player resulted in the same payoff value of 3**
  - **Distinguishes game as a *pure strategy* game**
  - **Outcome of 3 results from a pure (or dominant) strategy; it is referred to as a *saddle point* or *equilibrium point***
    - » **a value that is simultaneously the minimum of a row and the maximum of a column**
  - **3 is the *value of the game* (the average or expected game outcome)**
- **Minimax criterion results in the optimal strategy for each player only if both players use it**

Another Example

A professional athlete and his agent are negotiating the athlete's contract with his team's general manager. The various outcomes of the game are organized into the payoff table below.

Athlete/Agent	General Manager Strategies		
Strategies	A	B	C
1	$50,000	$35,000	$30,000
2	60,000	40,000	20,000

Another Example

- **Maximin strategy for Athlete/agent**
  - **Optimal strategy is strategy 1**

Athlete/Agent	General Manager Strategies		
Strategies	A	B	C
1	$50,000	$35,000	$30,000
2	60,000	40,000	20,000

Another Example

- **Minimax strategy for General Manager**
  - **Optimal strategy is strategy C**

Athlete/Agent Strategies	General Manager Strategies A	B	C
1	$50,000	$35,000	$30,000
2	60,000	40,000	20,000

A Mixed Strategy Game

- **If both players are logical and rational, it can be assumed a minimax criterion will be employed**
- **Existence of a saddle point is indicative of a pure strategy game**
- **A *mixed strategy game* results if:**
  - **Minimax criterion are not employed, or**
  - **Each player selects an optimal strategy and they *do not result in a saddle point* when the minimax criterion is used**
    - » **each player will play each strategy for a certain percentage of the time**

Mixed Strategy Solution

minimum of the maximum values

	Y_1	Y_2
X_1	**4**	**2**
X_2	1	10

maximum of the minimum values

- **No saddle point exists**
  - **Therefore not a pure strategy game**
- **This condition will not result in any dominant strategy for either player; instead a *closed loop* exists**
  - **Player X maximizes his gain by choosing strategy X_1; Player Y selects strategy Y_1 to minimize player X's gain**
  - **As soon as Player Y notices that Player X was using strategy X_1, he switches to strategy Y_2**
  - **Player X then switches to strategy X_2**

Mixed Strategy Games

- For 2 X 2 games, an algebraic approach based on the diagram below can be used to determine the percentage of the time that each strategy will be played

	Y_1	Y_2	
X_1	4	2	*Q*
X_2	1	10	*1 - Q*
	P	*1 - P*	

Q and *1 - Q* = the fraction of time *X* plays strategies X_1 and X_2, respectively

P and *1 - P* = the fraction of time *Y* plays strategies Y_1 and Y_2, respectively

Mixed Strategy Games

- **Each player's overall objective is to determine the fraction of time that each strategy should be played in order to maximize winnings**
  - **A strategy that results in maximum winnings no matter what the other player's strategy happens to be**
  - **Best mixed strategy is found by equating a player's expected winnings for one of the opponents strategies with the expected winnings for the opponent's other strategy**
    - » **the *expected gain and loss method***
    - » **a plan of strategies such that the *expected gain* of the maximizing player or the *expected loss* of the minimizing player will be the same regardless of the opponent's strategy**

Mixed Strategy Games

- **Steps for determining the optimum mixed strategy for a 2 X 2 game algebraically**

  (1) Compute the expected gain for player X

  - **Arbitrarily assume that player Y selects strategy Y_1**
    - **given this condition, there is a probability *q* that player X selects strategy X_1 and a probability 1 - *q* that player X selects strategy X_2**
    - **expected gain = $4q + 1(1 - q) = 1 + 3q$**

  - **Arbitrarily assume that player Y selects strategy Y_2**
    - **given this condition, there is a probability *q* that player X selects strategy X_1 and a probability 1 - *q* that player X selects strategy X_2**
    - **expected gain = $2q + 10(1 - q) = 10 - 8q$**

Mixed Strategy Games

(2) Player X is indifferent to player Y's strategy

- Equate the expected gain from each of the strategies

$$1 + 3q = 10 - 8q \quad \rightarrow \quad 11q = 9; \;\; q = 9/11$$

q = the *percentage of time* that strategy X_1 is used

Player X's plan is to use strategy X_1 9/11 of the time and strategy X_2 2/11 of the time

Mixed Strategy Games

(3) Compute the expected loss for player Y

- Arbitrarily assume that player X selects strategy X_1
  - given this condition, there is a probability *p* that player Y selects strategy Y_1 and a probability 1 - *p* that player Y selects strategy Y_2
  - expected loss = $4p + 2(1 - p) = 2 + 2p$

- Arbitrarily assume that player X selects strategy X_2
  - given this condition, there is a probability *p* that player Y selects strategy Y_1 and a probability 1 - *p* that player Y selects strategy Y_2
  - expected loss = $1p + 10(1 - p) = 10 - 9p$

Mixed Strategy Games

(4) Player Y is indifferent to player X's strategy

– Equate the expected gain from each of the strategies

$2 + 2p = 10 - 9p$ → $11p = 8$; $p = 8/11$

p = the *percentage of time* that strategy Y_1 is used

Player Y's plan is to use strategy Y_1 8/11 of the time and strategy Y_2 3/11 of the time

Value of a Mixed Strategy Game

	Y_1	Y_2	
X_1	4	2	*9/11*
X_2	1	10	*2/11*
	8/11	*3/11*	

- Once optimum strategies are determined, the value of the game can be calculated by multiplying each game outcome times the fraction of time that each strategy is employed

Game Outcome	Q	P	
4	x 9/11	x 8/11	= 2.38
2	x 9/11	x 3/11	= 0.45
1	x 2/11	x 8/11	= 0.13
10	x 2/11	x 3/11	= 0.50
Value of the Game			3.46

  - Value of the game is the average or expected game outcome after a large number of plays

Value of a Mixed Strategy Game (A Shortcut)

- **Since optimal strategies are computed by equating the expected gains of both strategies for each player, the value of the game can be computed by multiplying game outcomes times their probabilities of occurrence for any row or column**

	Y_1	Y_2	
X_1	4	2	*9/11*
X_2	1	10	*2/11*
	8/11	*3/11*	

Value of the Game

- **Row 1: 4(8/11) + 2(3/11) = 38/11**
- **Row 2: 1(8/11) + 10(3/11) = 38/11**
- **Column 1: 4(9/11) + 1(2/11) = 38/11**
- **Column 2: 2(9/11) + 10(2/11) = 38/11**

Dominance

- **The principle of *dominance* can be used to reduce the size of games by eliminating strategies that would never be used**
  - **A strategy is *dominated*, and can therefore be eliminated, if all of its payoffs are worse or no better than the corresponding payoffs for another strategy**
    - » **playing an alternative strategy always yields an equal gain or better**

Dominance

	Y_1	Y_2
X_1	4	3
X_2	2	20
X_3	1	1

- X_3 will never be played because player X can always do better by playing X_1 or X_2

	Y_1	Y_2	Y_3	Y_4
X_1	-5	4	6	-3
X_2	-2	6	2	-20

Dominance

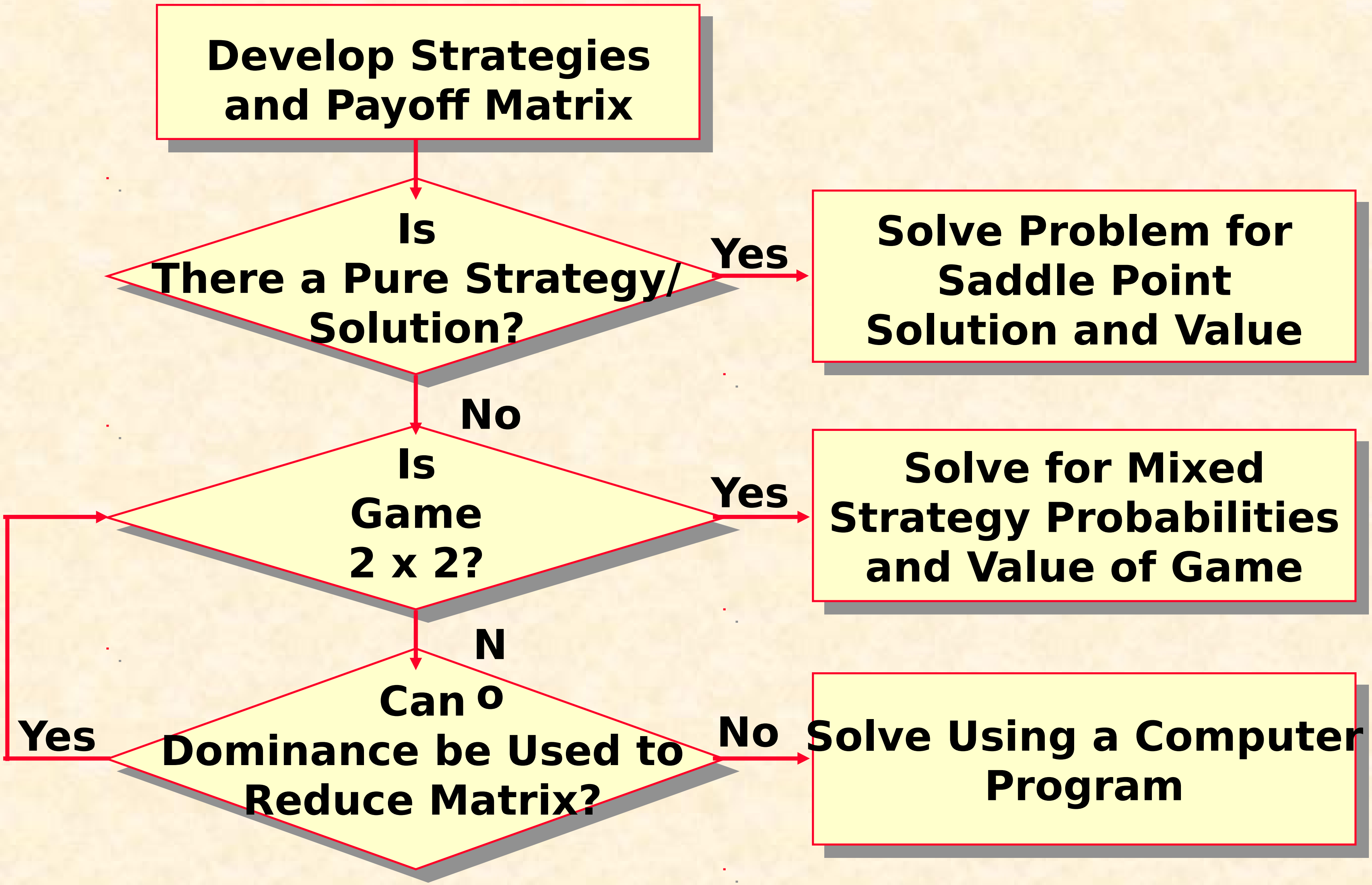
	Y_1	Y_2
X_1	4	3
X_2	2	20
~~X_3~~	~~1~~	~~1~~

- X_3 will never be played because player X can always do better by playing X_1 or X_2

	Y_1	~~Y_2~~	~~Y_3~~	Y_4
X_1	-5	~~4~~	~~6~~	-3
X_2	-2	~~6~~	~~2~~	-20

- Y_2 and Y_3 will never be played because player Y can always do better by playing Y_1 or Y_4

Solution Strategy



Solve for Saddle Point

- **Apply the maximin decision criterion for offensive player**
- **Apply the minimax decision criterion for defensive player**

Solve for Mixed Strategy Probabilities and Value of Game

- **If no saddle point exists, use expected gain and loss method to solve for mixed strategy probabilities and value of the game**

Another Example

Coloroid Camera Co. (company 1) plans to introduce a new instant camera and hopes to capture a large increase in its market share. Camco Camera Co. (company 2) hopes to minimize Coloroid's market share increase. The two companies dominate the camera market; any gain by Coloroid comes at Camco's expense. The payoff table, which includes the strategies and outcomes for each company, is shown below.

Company 1	Company 2 Strategies		
Strategies	A	B	C
1	9	7	2
2	11	8	4
3	4	1	7

Solution

Company 1 Strategies	Company 2 Strategies A	B	C
1	9	7	2
2	11	8	4
3	4	1	7

- **(1) Check for a pure strategy**
  - **maximin = 4 using strategy 2**
  - **minimax = 7 using strategy C**

no pure strategy exists

Solution

Company 1 Strategies	Company 2 Strategies A	B	C	row minimum
1	9	7	2	2
2	11	8	4	4
3	4	1	7	1
column maximum	11	8	7	

No Pure Strategy

Solution

Company 1 Strategies	Company 2 Strategies A	B	C
1	9	7	2
2	11	8	4
3	4	1	7

- **(2) Is game 2 X 2?**

  No

Solution

- (3) Check for dominance
  - 2 dominates 1; B dominates A

Company 1 Strategies	Company 2 Strategies A	B	C
1	9	7	2
2	11	8	4
3	4	1	7

Company 1 Strategies	Company 2 Strategies B	C
2	8	4
3	1	7

Solution

- **(4) Solve for Mixed Strategy Probabilities and Value of Game using expected gain and loss method**

<u>Compute the expected gain for company 1</u>

- **Arbitrarily assume that company 2 selects strategy B**
- **Given this condition, there is a probability q that company 1 selects strategy 2 and a probability $1 - q$ that company 1 selects strategy 3**
  - » **expected gain = $8q + 1(1 - q) = 1 + 7q$**
- **Arbitrarily assume that company 2 selects strategy C**
- **Given this condition, there is a probability q that company 1 selects strategy 2 and a probability $1 - q$ that company 1 selects strategy 3**
  - » **expected gain = $4q + 7(1 - q) = 7 - 3q$**

Solution

- **Company 1 is indifferent to company 2's strategy**
  - **equate the expected gain from each of the strategies**

$$1 + 7q = 7 - 3q \qquad \rightarrow \qquad 10q = 6;\ \ q = .6$$

<u>*q* = the *percentage of time* that strategy 2 is used</u>

Company 1's plan is to use strategy 2 60% of the time and strategy 3 40% of the time

Solution

Company 1 Strategies	Company 2 Strategies B	C	
2	8	4	*0.6*
3	1	7	*0.4*

- **Expected gain (market share increase) can be computed using the payoff of either strategy B or C since the gain is equal for both**

- **EG(company 1) = .6(4) + .4(7) = 5.2% increase in market share**

Solution

Repeat for company 2

- **Arbitrarily assume that company 1 selects strategy 2**
- **Given this condition, there is a probability p that company 2 selects strategy B and a probability $1 - p$ that company 2 selects strategy C**
  - » **expected gain = $8p + 4(1 - p) = 4 + 4p$**
- **Arbitrarily assume that company 1 selects strategy 3**
- **Given this condition, there is a probability p that company 1 selects strategy B and a probability $1 - p$ that company 1 selects strategy C**
  - » **expected gain = $1p + 7(1 - p) = 7 - 6p$**

Expected Gain and Loss Method

– **Company 2 is indifferent to company 1's strategy**

» **equate the expected gain from each of the strategies**

$$4 + 4p = 7 - 6p \qquad 10p = 3; \quad p = .3$$

p **= the *percentage of time* that strategy B is used**

Company 2's plan is to use strategy B 30% of the time and strategy C 70% of the time

Mixed Strategy Solution

Company 1 Strategies	Company 2 Strategies B	C	
2	8	4	*0.6*
3	1	7	*0.4*
	0.3	*0.7*	

- **Expected loss (market share decrease) can be computed using the payoff of either strategy 1 or 2 since the gain is equal for both**

- **EL(company 2) = .3(8) + .7(4) = 5.2% decrease in market share**

Mixed Strategy Summary

Company 1

Strategy 2: 60% of the time

Strategy 3: 40% of the time

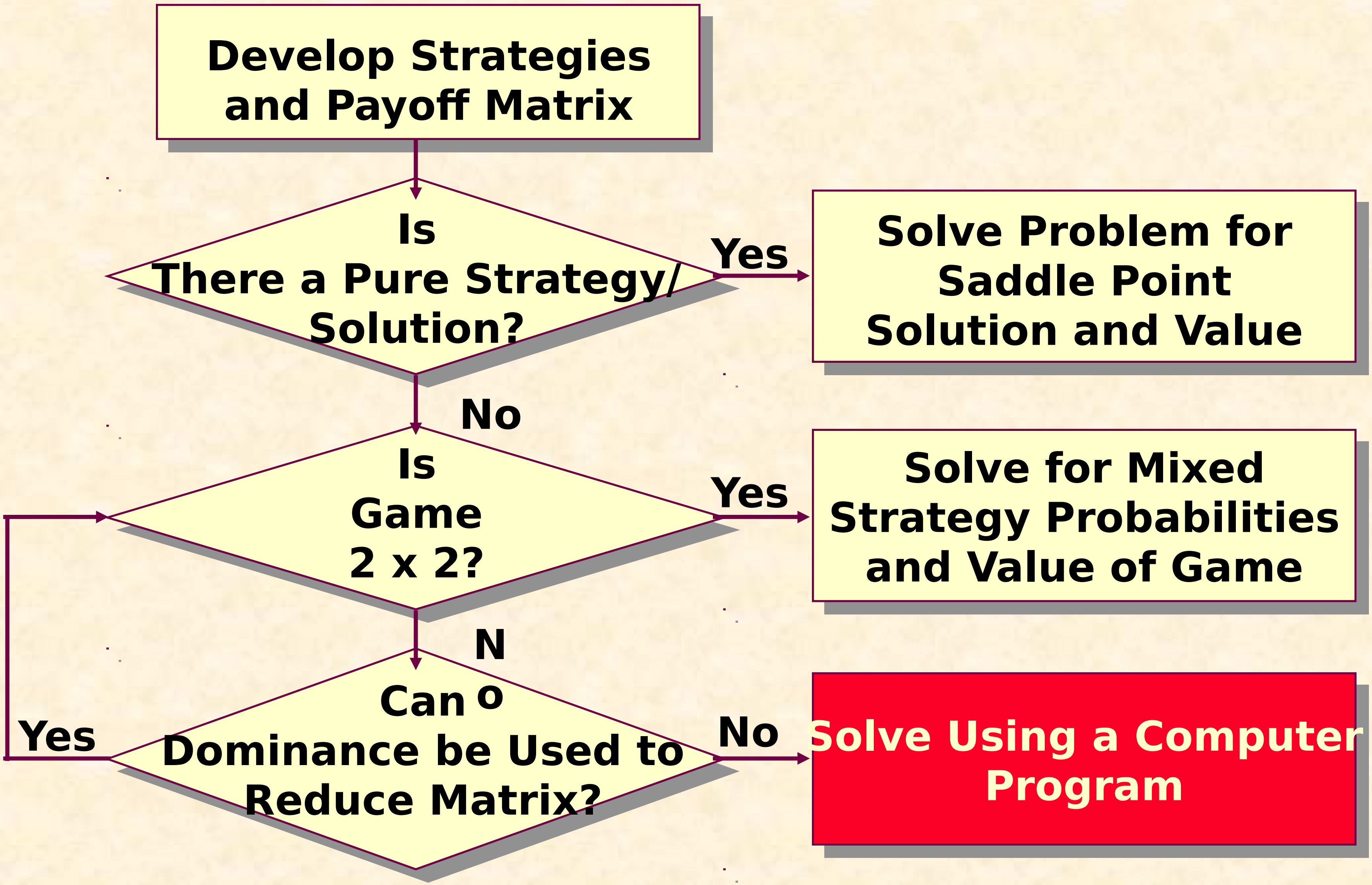
Company 2

Strategy B: 30% of the time

Strategy C: 70% of the time

- **The expected gain for company 1 is 5.2% of the market share and the expected loss for company 2 is 5.2 % of the market share**
  - **Mixed strategies for each company have resulted in an equilibrium point such that a 5.2% expected gain for company 1 results in a simultaneous 5.2% loss for company 2**
  - **How does this compare with the maximin/minimax strategies?**

Solution Strategy



Computer Solutions: QM for Windows

Game Theory

File Edit View Module Tables Window Help

Print Screen Edit Data

Instruction: There are more results available in additional windows. These may be opened by double clicking or using the WINDOW option in the Main Menu.

Game Theory Results

QM for Windows - Example 1 Solution

	A	B	C	Row Mix
1	9.	7.	2.	0.
2	11.	8.	4.	0.6
3	4.	1.	7.	0.4
Column Mix--->	0.	0.3	0.7	
Value of game (to	5.2			

Row's Expected Values

QM for Windows - Example 1 Solution

	Col mix 1 * cell payoff	Col mix 2 * cell payoff	Col mix 3 * cell payoff	Expected Value (row
Column's Optimal Mix	0.	0.3	0.7	
1	0.	2.1	1.4	3.5
2	0.	2.4	2.8	5.2
3	0.	0.3	4.9	5.2
Value of game (to				5.2

Column's Expected Values

QM for Windows - Example 1 Solution

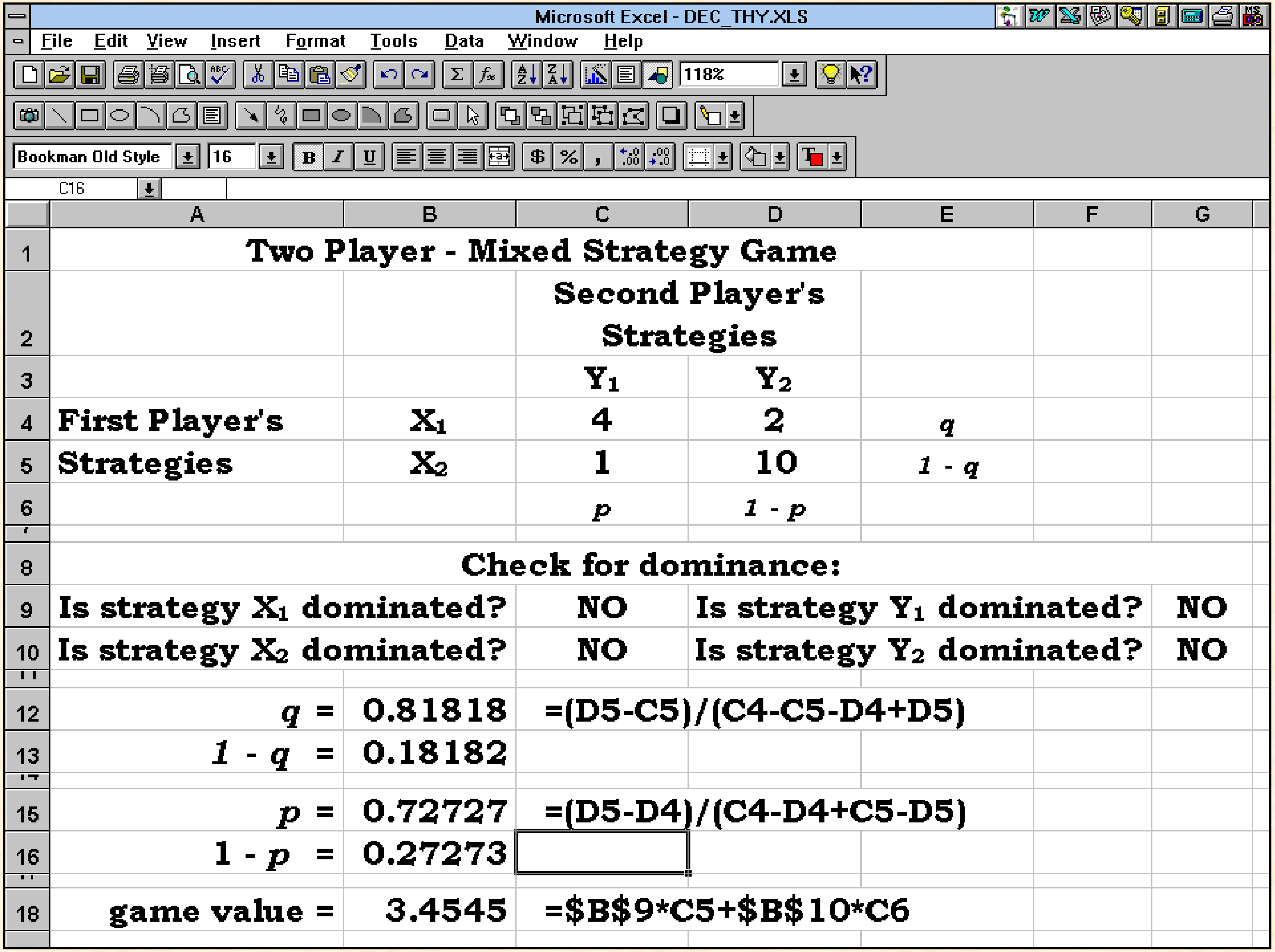
	Optimal Row Mix	A	B	C
Row 1 mix * cell payoff	0.	0.	0.	0.
Row 2 mix * cell payoff	0.6	6.6	4.8	2.4
Row 3 mix * cell payoff	0.4	1.6	0.4	2.8
Expected Value (Col		8.2	5.2	5.2
Value of game (to row)	5.2			

Maximin & Minimax

QM for Windows - Example 1 Solution

	A	B	C	Row Minimum	Maximin
1	9.	7.	2.	2.	
2	11.	8.	4.	4.	4.
3	4.	1.	7.	1.	
Column	11.	8.	7.		
Minimax			7.		
4 <= value <= 7					

Computer Solutions: Excel



	A	B	C	D	E	F	G
1	Two Player - Mixed Strategy Game						
2			Second Player's Strategies				
3			Y_1	Y_2			
4	First Player's	X_1	4	2	*q*		
5	Strategies	X_2	1	10	*1 - q*		
6			*p*	*1 - p*			
7							
8	Check for dominance:						
9	Is strategy X_1 dominated?		NO	Is strategy Y_1 dominated?			NO
10	Is strategy X_2 dominated?		NO	Is strategy Y_2 dominated?			NO
11							
12	*q* =	0.81818	=(D5-C5)/(C4-C5-D4+D5)				
13	*1 - q* =	0.18182					
14							
15	*p* =	0.72727	=(D5-D4)/(C4-D4+C5-D5)				
16	1 - *p* =	0.27273					
17							
18	game value =	3.4545	=$B$9*C5+$B$10*C6				